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On a generalisation of totally permutable products of finite groups

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ABSTRACT

A group $G = AB$ is a weakly totally permutable product of subgroups A and B if for every subgroup, U of A such that $U \leqslant A \cap B$ or $A \cap B \leqslant U$, permutes with every subgroup of B and if for every subgroup V of B such that $V \leqslant A \cap B$ or $A \cap B \leqslant V$, permutes with every subgroup of A. Results on totally permutable products are extended to weakly totally permutable products. In particular it is shown that for a weakly totally permutable product if the factors are in $\mathcal{F}$, then the product is also in $\mathcal{F}$, where $\mathcal{F}$ is a formation containing $\mathcal{U}$, the class of finite supersoluble groups. It is also shown that $\mathcal{F}$-residuals and $\mathcal{F}$-projectors behave nicely in weakly totally permutable products when $\mathcal{F}$ is a saturated formation containing $\mathcal{U}$.

1. Introduction

All groups considered are finite.

A group $G = AB$ is the mutually permutable product of subgroups A and B if A permutes with every subgroup of B and B permutes with every subgroup of A. A group $G = AB$ is the totally permutable product of subgroups A and B if every subgroup of A permutes with every subgroup of B. These types of products of groups were introduced by Asaad and Shaalan in [4]. Asaad and Shaalan [4] proved that a totally permutable product of two supersoluble groups is also supersoluble. In the same paper they showed that a mutually permutable product of two supersoluble subgroups is not supersoluble in general. This gave rise to a study of totally permutable products and mutually permutable products in the framework of formation theory [1,2,5,7–10,12,15] and Fitting classes [13,14, 17–19]. R. Maier [20] generalised Asaad and Shaalan's result to any saturated formation containing the class $\mathcal{U}$, of finite supersoluble groups. Ballester-Bolinches and Pérez-Ramos [9] extended Maier's result to non-saturated formations containing $\mathcal{U}$:

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Theorem A. *(See [9, Theorem 1].) Let a group $G = AB$ be the totally permutable product of subgroups A and B. Let $\mathcal{F}$ be a formation containing $\mathcal{U}$. If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $G \in \mathcal{F}$.*

In [7] Ballester-Bolinches et al. proved that $\mathcal{F}$-projectors and $\mathcal{F}$-residuals behave nicely in a totally permutable product when $\mathcal{F}$ is a saturated formation containing $\mathcal{U}$:

Theorem B. *(See [7, Theorem A].) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Let a group $G = AB$ be the totally permutable product of subgroups A and B. Then $G^{\mathcal{F}} = A^{\mathcal{F}} B^{\mathcal{F}}$.*

Theorem C. *(See [7, Theorem B].) Let the group $G = AB$ be the totally permutable product of the subgroups A and B. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If A_1 and B_1 are $\mathcal{F}$-projectors of A and B respectively, then A_1B_1 is an $\mathcal{F}$-projector of G.*

Peter Hauck defined another type of product of subgroups:

Definition. A group $G = AB$ is a weakly totally permutable product of subgroups A and B if for every subgroup, U of A such that $U \leqslant A \cap B$ or $A \cap B \leqslant U$, permutes with every subgroup of B and if for every subgroup V of B such that $V \leqslant A \cap B$ or $A \cap B \leqslant V$, permutes with every subgroup of A.

A totally permutable product is a weakly totally permutable product and a weakly totally permutable product is mutually permutable. However the converse of this statement is not necessarily true (cf., the remark after Example 1).

If mutually permutable products are viewed as a generalisation of products of normal subgroups and totally permutable products as a generalisation of central products, then weakly totally permutable products fit into this picture as a generalisation of products $G = AB$ with normal subgroups A and B such that $A \cap B \leqslant \zeta(G)$, the centre of G. The natural question is to ask if results on totally permutable products can be extended to weakly totally permutable products.

In this article weakly totally permutable products are studied in the framework of formation theory. In particular we extend Theorem A, Theorem B and Theorem C to weakly totally permutable products:

Theorem 1. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $G^{\mathcal{F}} = A^{\mathcal{F}} B^{\mathcal{F}}$.*

Theorem 2. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. If A_1 and B_1 are $\mathcal{F}$-projectors of A and B respectively, then A_1B_1 is an $\mathcal{F}$-projector of G.*

Theorem 3. *Let $\mathcal{F}$ be a formation containing $\mathcal{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. If A and B belong to $\mathcal{F}$, then G also belongs to $\mathcal{F}$.*

2. Preliminary results

Lemma 1. *(See [9, Lemma 1].) Let the group $G = NB$ be the product of subgroups N and B. Suppose that N is normal in G. Since B acts by conjugation on N, we can construct the semidirect product $X = [N]B$, with respect to this action. Then the natural map $\alpha : X \to G$ given by $(nb)\alpha = nb$, for every $n \in N$ and every $b \in B$, is an epimorphism, $\mathrm{Ker}\,\alpha \cap N = 1$ and $\mathrm{Ker}\,\alpha \leqslant C_X(N)$.*

Lemma 2. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B.*

(i) *If H and K are subgroups of G such that $A \cap B \leqslant H \leqslant A$ and $A \cap B \leqslant K \leqslant B$, then the subgroup HK is a weakly totally permutable product of subgroups H and K.*

(ii) *Then $A \cap B$ is a nilpotent subnormal subgroup of G.*
(iii) *If N is a minimal normal subgroup of G such that $N \leqslant A \cap B$, then N is a cyclic group of order p for some prime p.*
(iv) *If A is a minimal normal subgroup of G, then $G = AB$ is the totally permutable product of subgroups A and B.*

Proof. For (i) we note that $H \cap K = A \cap B$. Let U be a subgroup of H such that $U \leqslant H \cap K = A \cap B$. Then U permutes with every subgroup of B and hence every subgroup of K.

Let V be a subgroup of H such that $H \cap K \leqslant V$. Note that V is a subgroup of A and $A \cap B = H \cap K \leqslant V$. It follows that V permutes with every subgroup of B and hence every subgroup of K. The same is true if K and H are interchanged. Hence the result follows.

For (ii) Let H be a subgroup of $A \cap B$. By definition H is a permutable subgroup and hence a subnormal subgroup in both A and B.

By [3, Theorem 7.5.7], H is a subnormal subgroup of $AB = G$. So H is a subnormal subgroup of $A \cap B$. It follows that $A \cap B$ is a nilpotent group.

For (iii) since $N \leqslant A \cap B$, a nilpotent group we have N is an abelian p-group for some prime p. Let P be a Sylow p-subgroup of G. Then $N \cap Z(P) \neq 1$. Let $g \in N \cap Z(P)$ be an element of order p. So the subgroup $\langle g\rangle$ is permutable in A and in B. Hence $\langle g\rangle$ is normalised by all Sylow q-subgroups of both A and B, that is, $\langle g\rangle$ is normal in A and B. Therefore $N = \langle g\rangle$ as required.

For (iv) by [11, Lemma 1(vii)] either $A \cap B = 1$ or $A \cap B = A$. If $A \cap B = 1$, then the result follows. If $A \cap B = A$, then $G = (A \cap B)B$ is the totally permutable product of subgroups $A \cap B$ and B. □

Lemma 2(ii) generalises [20, Lemma 2(b)]. Totally and mutually permutable products behave nicely with respect to factor groups. However weakly totally permutable products do not have this property as the following example shows:

Example 1. Let G be the direct product of C_3 (cyclic group of order 3) with B, where B is an extraspecial group of order 27 and exponent 3 whose presentation is

$$B = \langle x, y \mid x^3 = y^3 = 1,\ [x, y] = z,\ zx = xz,\ yz = zy\rangle.$$

Suppose $C_3 = \langle c\rangle$. Let $A = \langle z, cx\rangle$. Then A and B are weakly totally permutable. But if $N = \langle c\rangle$, AN/N and BN/N are not weakly totally permutable since $\langle y\rangle N \leqslant AN/N \cap BN/N$ does not permute with $\langle x\rangle N = \langle cx\rangle N \leqslant AN/N \cap BN/N$.

Remark. In Example 1, the G/N is a mutually permutable product of subgroups AN/N and BN/N of G/N, but AN/N and BN/N are not weakly totally permutable. The group $G = AB$ is a weakly totally permutable product of subgroups but subgroups A and B are not totally permutable.

The next result shows that if the normal subgroup is a product of normal subgroups of G contained in the factors then weakly totally permutable products behave nicely with respect to factor groups.

Lemma 3. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B.*

(i) *If M and N are normal subgroups of G such that $M \leqslant A$ and $N \leqslant B$, then $G/MN = (AN/MN)(BM/MN)$ is the weakly totally permutable product of subgroups AN/MN and BM/MN.*
(ii) *Let N be a normal subgroup of G such that $(N \cap A)(N \cap B) \neq 1$. Suppose $X = N \cap A$ and $Y = N \cap B$. Then $G/XY = (AY/XY)(BX/XY)$ is the weakly totally permutable product of subgroups AY/XY and BX/XY.*

Proof. For (i) we note that $AN/MN \cap BM/MN = (AN \cap BM)/MN$. Since $M \leqslant A$ and $N \leqslant B$ we have $(AN \cap BM)/MN = (AN \cap B)M/MN = (A \cap B)MN/MN$ by Dedekind's Identity. Let HMN/MN be a subgroup of AN/MN and let KMN/MN be a subgroup of BM/MN. Then there is a subgroup W of B such that $WMN/MN = KMN/MN$.

If HMN/MN is a subgroup of $(A \cap B)MN/MN$, then there is a subgroup V of $A \cap B$ such that $VMN/MN = HMN/MN$. Since V permutes with W we have that HMN/MN permutes with KMN/MN. Suppose HMN/MN contains $(A \cap B)MN/MN$. Then there is a subgroup U of A such that $UMN/MN = HMN/MN$. Consider $Y = U(A \cap B)$. Then $YMN/MN = U(A \cap B)MN/MN = HMN/MN$. Since Y permutes with W we also have HMN/MN permutes with KMN/MN.

The same is true if H and A are interchanged with K and B. Hence the result follows.

For (ii) we have XY is a normal subgroup of G by [11, Lemma 1(ii)]. Then $AY/XY \cap BX/XY = (AY \cap BX)/XY = (A \cap B)XY/XY$. The result now follows arguing as in the proof of (i). □

Lemma 4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Let N be a minimal normal subgroup of G of order p for some prime p. Then G belongs to $\mathcal{F}$ if and only if G/N belongs to $\mathcal{F}$.*

Proof. If G belongs to $\mathcal{F}$, then G/N also belongs to $\mathcal{F}$. Suppose $G/N \in \mathcal{F}$. Since N is a cyclic group of order p, $G/C_G(N)$ is abelian of exponent $p-1$. Hence $G/C_G(N) \in U(p) \subseteq F(p)$, where U and F are canonical local definitions of $\mathcal{U}$ and $\mathcal{F}$, respectively. Let H/K be a chief factor of G such that $N \leqslant K < H$. Then since G/N belongs to $\mathcal{F}$ we have that $G/C_G(H/K) \in F(q)$ for primes q dividing $|H/K|$. Hence G belongs to $\mathcal{F}$. □

The following result extends Maier's result [20, Theorem] and its converse [9, Lemma 4] to weakly totally permutable products:

Lemma 5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then G belongs to $\mathcal{F}$ if and only if A and B belong to $\mathcal{F}$.*

Proof. By Theorem 4.5.8 of [6] if $(A \cap B)_G = 1$, then G belongs to $\mathcal{F}$ if and only if A and B belong to $\mathcal{F}$. So we may assume that there is a minimal normal subgroup of G such that $N \leqslant (A \cap B)_G$. By Lemma 2(iii) N is cyclic of order p for some prime p. Arguing by induction on $|G|$ and by Lemma 3(i) we have G/N belongs to $\mathcal{F}$ if and only if A/N and B/N belong to $\mathcal{F}$. Since N is a minimal normal subgroup of both subgroups A and B we have G belongs to $\mathcal{F}$ if and only if A and B belong to $\mathcal{F}$ by Lemma 4 as required. □

Lemma 6. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. If A_1 and B_1 are $\mathcal{F}$-projectors of A and B respectively, then $A \cap B$ is a subgroup of both A_1 and B_1.*

Proof. Let A_1 be an $\mathcal{F}$-projector of A. Then $H = A_1(A \cap B)$ is the totally permutable product of subgroups A_1 and $A \cap B$. Since $A \cap B$ is nilpotent, we have $H \in \mathcal{F}$ by Lemma 5. Since A_1 is $\mathcal{F}$-maximal in A we have $H = A_1$ and hence $A \cap B$ is a subgroup of A_1.

Analogously B_1 contains $A \cap B$. □

Lemma 7. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $A^{\mathcal{F}}, B^{\mathcal{F}} \leqslant G^{\mathcal{F}}$.*

Proof. Suppose the lemma is not true and let G be a minimal counterexample. Let $M = (G^{\mathcal{F}} \cap A)(G^{\mathcal{F}} \cap B) \neq 1$. By Lemma 3(ii) we have G/M is the weakly totally permutable product of subgroups AM/M and BM/M. By the choice of G we have $A^{\mathcal{F}}M \leqslant G^{\mathcal{F}}M = G^{\mathcal{F}}$ a contradiction.

Suppose $M = 1$. Let H/K be a chief factor of G such that $1 \leqslant K < H \leqslant G^{\mathcal{F}}$. Since G/K is the weakly totally permutable product of subgroups AK/K and BK/K and $H/K \cap AK/K = H/K \cap BK/K = 1$ it follows that $|H/K| = p$ for a prime p by [11, Lemma 2]. Hence $G/C_G(H/K) \in U(p) \subseteq F(p)$, where U and F are canonical local definitions of $\mathcal{U}$ and $\mathcal{F}$, respectively. Let H/K be a chief factor of G such that $G^{\mathcal{F}} \leqslant K < H$. Then since $G/G^{\mathcal{F}}$ belongs to $\mathcal{F}$ we have that $G/C_G(H/K) \in F(q)$ for primes q dividing $|H/K|$. Hence G belongs to $\mathcal{F}$ and by Lemma 5 both A and B belong to $\mathcal{F}$ our final contradiction.

The same is true if A and B are interchanged. □

Lemma 8. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. If B is supersoluble, then $G^{\mathcal{U}} = A^{\mathcal{U}}$.*

Proof. Suppose the lemma is not true and let G be a minimal counterexample. So we may assume that $G^{\mathcal{U}}N = A^{\mathcal{U}}N$ for any minimal normal subgroup of G such that $N \leqslant A$ or $N \leqslant B$. By [6, Theorem 4.5.8] we have that if $(A \cap B)_G = 1$, then $G^{\mathcal{U}} = A^{\mathcal{U}}$. We may assume that there is a minimal normal subgroup N of G contained in $(A \cap B)_G$. By Lemma 3(i) and the choice of G we have $G^{\mathcal{U}}N = A^{\mathcal{U}}N$. It follows that $G^{\mathcal{U}} \leqslant A$ and hence $A^{\mathcal{U}}$ is normal subgroup of $G^{\mathcal{U}}$ by Lemma 7. Let M be a minimal normal subgroup of G contained in $G^{\mathcal{U}}$. By Lemma 3(i) $G^{\mathcal{U}}M = G^{\mathcal{U}} = A^{\mathcal{U}}M$.

Let $A^{\mathcal{S}} \neq 1$ be the soluble residual of A. By [11, Corollary 3] $A^{\mathcal{S}}$ is a normal subgroup of G. Since $A/A^{\mathcal{U}} \in \mathcal{U} \subseteq \mathcal{S}$ we have $A^{\mathcal{S}} \leqslant A^{\mathcal{U}}$. Hence $G^{\mathcal{U}} = A^{\mathcal{S}}A^{\mathcal{U}} = A^{\mathcal{U}}$ which is a contradiction.

So A is soluble. Since B is supersoluble we have G is soluble by [11, Corollary 2]. Hence M is abelian. Now $G^{\mathcal{U}}/A^{\mathcal{U}} = MA^{\mathcal{U}}/A^{\mathcal{U}} \cong M/(M \cap A^{\mathcal{U}})$ is abelian. It follows that $(G^{\mathcal{U}})' \leqslant A^{\mathcal{U}}$. If $(G^{\mathcal{U}})' \neq 1$, then $G^{\mathcal{U}} = A^{\mathcal{U}}(G^{\mathcal{U}})' = A^{\mathcal{U}}$ because $(G^{\mathcal{U}})'$ is a normal subgroup of G. So $G^{\mathcal{U}}$ must be abelian.

Let A_1 be a $\mathcal{U}$-projector of A. Then $A \cap B \leqslant A_1$ by Lemma 6 and $G = A^{\mathcal{U}}A_1B = G^{\mathcal{U}}(A_1B)$. Since A_1B is the weakly totally permutable product of subgroups A_1 and B we have A_1B is supersoluble by Lemma 5. Let F be a $\mathcal{U}$-maximal subgroup G such that $A_1B \leqslant F$. Since $G = G^{\mathcal{U}}F$, F is a $\mathcal{U}$-projector of G by [16, III, Lemma (3.14)]. By [16, IV, Theorem (5.18)] we have $G^{\mathcal{U}} \cap F = 1$. Hence $G^{\mathcal{U}} = A^{\mathcal{U}}(A_1B \cap G^{\mathcal{U}}) = A^{\mathcal{U}}$ which is our final contradiction. □

Lemma 9. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then*

$$\left[A, B^{\mathcal{U}}\right] = \left[A^{\mathcal{U}}, B\right] = 1.$$

Proof. Suppose the lemma is not true and let (G, B) be a counterexample with $|G| + |B|$ minimal. We argue that B is supersoluble. Let $D = A \cap B$. If for all $x \in B$ we have $D\langle x\rangle$ is a proper subgroup of G, then by the choice of G, $B = \langle D\langle x\rangle \mid x \in B\rangle$ centralises $A^{\mathcal{U}}$ a contradiction. So we may assume that $B = D\langle x\rangle$. Since D is nilpotent and $\langle x\rangle$ is cyclic we have B is supersoluble by Lemma 5.

By Lemma 8 we have $G^{\mathcal{U}} = A^{\mathcal{U}}$. Let N be a minimal normal subgroup of G contained in $A^{\mathcal{U}}$. Then $G/N = (A/N)(BN/N)$ is the weakly totally permutable product of subgroups A/N and BN/N by Lemma 3(i). Since BN/N is supersoluble, we have $BN/N \leqslant C_{G/N}(G^{\mathcal{U}}N/N)$ by the choice of G and $[B, A^{\mathcal{U}}] \leqslant N$.

If $A \cap B = B$, then $A = G$ and B are totally permutable and a contradiction follows from [9, Lemma 8]. Let M be a maximal subgroup of B such that $A \cap B \leqslant M$. Then AM is the weakly totally permutable product of subgroups A and M. So $(AM)^{\mathcal{U}} = A^{\mathcal{U}}$. Also $M \leqslant C_G(A^{\mathcal{U}})$ by the choice of (G, B). If M_1 and M_2 are maximal subgroups of B such that $A \cap B \leqslant M_1$ and $A \cap B \leqslant M_2$, then $B = \langle M_1, M_2\rangle \leqslant C_G(A^{\mathcal{U}})$ which is a contradiction. So B has a unique maximal subgroup M such that $A \cap B \leqslant M$ and $M \leqslant C_G(A^{\mathcal{U}})$.

Let A_1 be a $\mathcal{U}$-projector of A. Then $A \cap B \leqslant A_1$. By Lemma 5 we have A_1B is supersoluble and $G = (A_1B)A^{\mathcal{U}}$. Let U be a $\mathcal{U}$-maximal subgroup of G containing A_1B. Then $G = UA^{\mathcal{U}}$. So $Z_{\mathcal{U}}(G) = C_U(A^{\mathcal{U}})$ by [16, IV, Theorem (6.14)], where $Z_{\mathcal{U}}(G)$ is the $\mathcal{U}$-hypercentre of G. This implies $M \leqslant Z_{\mathcal{U}}(G)$. We now want to show that $N \leqslant Z_{\mathcal{U}}(G)$.

By [6, Theorem 4.4.5] B is a normal subgroup of $BA^{\mathcal{U}}$ and so $[B, A^{\mathcal{U}}] \leqslant B$. Suppose $N \cap B = 1$. Then $[B, A^{\mathcal{U}}] \leqslant B \cap N = 1$, a contradiction. Hence $N \leqslant A \cap B \leqslant M \leqslant Z_{\mathcal{U}}(G)$ and $Z_{\mathcal{U}}(G/N) = Z_{\mathcal{U}}(G)/N$.

But $G/N = (UN/N)(A^{\mathcal{U}}/N)$ and $(G/N)^{\mathcal{U}} = A^{\mathcal{U}}/N$. Let U_1/N be a $\mathcal{U}$-maximal subgroup of G/N containing UN/N. We have $Z_{\mathcal{U}}(G)/N = Z_{\mathcal{U}}(G/N) = C_{U_1/N}(A^{\mathcal{U}}/N)$ by [16, IV, Theorem (6.14)]. By the choice of G we also have that $B/N \leqslant Z_{\mathcal{U}}(G)/N$ and so $B \leqslant Z_{\mathcal{U}}(G)$ which is our final contradiction.

The same is true if A and B are interchanged. □

Lemma 9 generalises [7, Corollary].

Lemma 10. *Let $\mathcal{F}$ be a formation containing $\mathcal{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $A^{\mathcal{F}}$ and $B^{\mathcal{F}}$ are normal subgroups of G.*

Proof. Since $A/A^{\mathcal{U}} \in \mathcal{U} \subseteq \mathcal{F}$, $A^{\mathcal{F}} \leqslant A^{\mathcal{U}}$. By Lemma 9, B centralises $A^{\mathcal{U}}$ and hence $B \leqslant C_G(A^{\mathcal{F}})$. Since $A^{\mathcal{F}}$ is normal in A we have $A^{\mathcal{F}}$ is a normal subgroup in G.

Analogously $B^{\mathcal{F}}$ is normal in G. □

3. Proof of main results

Proof of Theorem 1. We prove the theorem by induction on $|G|$. By Lemma 5 we may assume that $G^{\mathcal{F}} \neq 1$ and $A^{\mathcal{F}}B^{\mathcal{F}} \neq 1$. Let N be a minimal normal subgroup of G contained in $A^{\mathcal{F}}$ using Lemma 10. Then $G/N = (A/N)(BN/N)$ is the weakly totally permutable product of subgroups A/N and BN/N and so $G^{\mathcal{F}}N = A^{\mathcal{F}}B^{\mathcal{F}}N = A^{\mathcal{F}}B^{\mathcal{F}}$. Hence $G^{\mathcal{F}} \leqslant A^{\mathcal{F}}B^{\mathcal{F}}$. By Lemma 7 $A^{\mathcal{F}}B^{\mathcal{F}} \leqslant G^{\mathcal{F}}$. Hence the result now follows. □

Proof of Theorem 2. Suppose the theorem is not true and let $G = AB$ be a minimal counterexample. So $A \neq 1$ and $B \neq 1$. By Lemma 5 and Lemma 10 we may assume that there exists a minimal normal subgroup N of G such that $N \leqslant A$. Then $G/N = (A/N)(BN/N)$ is the weakly totally permutable product of subgroups A/N and BN/N. We have that $(A_1N/N)(B_1N/N) = (A_1B_1)N/N$ is an $\mathcal{F}$-projector of G/N by the choice of G. Consider $C = (A_1B_1)N$. Suppose C is a proper subgroup of G. Since $A \cap B$ is contained in both A_1 and B_1, the subgroup $(A_1N)B_1$ is the weakly totally permutable product of subgroups A_1N and B_1. Also A_1 is an $\mathcal{F}$-projector of A_1N by [16, III, Lemmas (3.14) and (3.18)]. This implies that A_1B_1 is an $\mathcal{F}$-projector of C by the minimality of G. Therefore A_1B_1 is an $\mathcal{F}$-projector of G using [16, III, Proposition (3.7)] which is a contradiction.

Hence $G = (A_1B_1)N$. By Theorem 1 $A_1B_1 \in \mathcal{F}$. Since $G/N \in \mathcal{F}$, $G^{\mathcal{F}} \leqslant N$. Also $G^{\mathcal{F}}$ is a normal subgroup of G so $G^{\mathcal{F}} = 1$ or $G^{\mathcal{F}} = N$. If $G^{\mathcal{F}} = 1$, then $A_1 = A$ and $B_1 = B$ by Theorem 1, a contradiction. Hence $G^{\mathcal{F}} = N$. Let F be an $\mathcal{F}$-maximal subgroup of G containing A_1B_1. It follows that F is an $\mathcal{F}$-projector of G by [16, III, Lemmas (3.14) and (3.18)]. Also $F = F \cap A_1B_1N = (A_1(F \cap N))B_1$ which is the weakly totally permutable product of subgroups $A_1(F \cap N)$ and B_1. Since $F \in \mathcal{F}$ we have that $A_1(F \cap N) \in \mathcal{F}$ by Theorem 1. But A_1 is $\mathcal{F}$-maximal in A, so $A_1 = A_1(F \cap N)$ and $F = (A_1(F \cap N))B_1 = A_1B_1$ is an $\mathcal{F}$-projector of G, our final contradiction. □

Proof of Theorem 3. Suppose the theorem is not true and let G be a counterexample with $|G| + |A| + |B|$ minimal. So $A, B \in \mathcal{F}$ but $G \notin \mathcal{F}$. By Lemma 5 either A or B is not supersoluble. We may assume $A^{\mathcal{U}} \neq 1$. By Lemma 9, B centralises $A^{\mathcal{U}}$. It follows that $A^{\mathcal{U}}$ centralises $\langle B^G \rangle$ since $A^{\mathcal{U}}$ is normal in G. Let A_1 be a $\mathcal{U}$-projector of A. Then $A \cap B \leqslant A_1$ by Lemma 6 and $A = A_1A^{\mathcal{U}}$. Consider A_1B, the weakly totally permutable product of subgroups A_1 and B. We have $A_1 \in \mathcal{U} \subseteq \mathcal{F}$ and $B \in \mathcal{F}$ and $A_1B \in \mathcal{F}$ since $|A_1B| + |A_1| + |B| < |G| + |A| + |B|$. Since $\langle B^G \rangle$ is a normal subgroup of G, A acts on $\langle B^G \rangle$ by conjugation. Let $Z = [\langle B^G \rangle]A$ be the semidirect product of $\langle B^G \rangle$ and A with respect to this action. So G is a quotient group of Z by Lemma 1. We want to show that Z belongs to $\mathcal{F}$. Since $A^{\mathcal{U}}$ centralises $\langle B^G \rangle$, $A^{\mathcal{U}}$ is normal in Z. So $Z/A^{\mathcal{U}} = [\langle B^G \rangle]A_1A^{\mathcal{U}}/A^{\mathcal{U}}$ is isomorphic to $[\langle B^G \rangle](A_1/(A_1 \cap A^{\mathcal{U}}))$ a quotient group of $[\langle B^G \rangle]A_1$. Since $A^{\mathcal{U}} \leqslant C_G(B)$ we have $\langle B^G \rangle = \langle B^A \rangle = \langle B^{A_1} \rangle$. Then there exists an epimorphism $\varphi : [\langle B^G \rangle]A_1 \to BA_1$ by Lemma 1. We have $[\langle B^G \rangle]A_1/\langle B^{A_1} \rangle$ which is isomorphic to $A_1 \in \mathcal{U} \subseteq \mathcal{F}$ and $[\langle B^G \rangle]A_1/\ker\varphi \cong BA_1 \in \mathcal{F}$. Hence $[\langle B^G \rangle]A_1/(\ker\varphi \cap \langle B^{A_1} \rangle) = [\langle B^G \rangle]A_1 \in \mathcal{F}$. Therefore $Z/A^{\mathcal{U}} \in \mathcal{F}$ and so $Z/\langle B^G \rangle \cong A \in \mathcal{F}$. By Lemma 1, $\langle B^G \rangle \cap A^{\mathcal{U}} = 1$ which implies $Z/(\langle B^G \rangle \cap A^{\mathcal{U}}) = Z \in \mathcal{F}$. The result follows since G is a quotient group of Z. □

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